DISCOVERING THE FUNDAMENTAL THEOREMS OF CALCULUS USING COMPUTER ALGEBRA SYSTEMS

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One of the most exciting developments in mathematics in recent years is the widespread availability of Computer Algebra Systems (CAS) or Symbolic Manipulation packages. These software packages provide us with the opportunity to use the computer in a symbolic rather than a computational mode and hence open the doors to a host of incredible possibilities. They have made major inroads into many areas of mathematical research, and many people have anticipated significant roles for such packages in mathematical education. To date, however, most of these educational efforts have not borne much fruit since there is considerable "overhead" involved (e.g., teaching students to use them and developing appropriate teaching strategies based on them) in using any of these systems.

In the present article, we will discuss some extensions of ideas first presented by Mathews [1],[2] which provide a dramatically new way to approach the entire notion of integration in introductory calculus. The ideas we will describe allow the students themselves to discover the antiderivative and the two Fundamental Theorems of Calculus in an especially simple and natural way.

The particular CAS system used there is muMATH or its successor DERIVE for the IBM PC or the Apple II series since they are probably the most widely available. We note that the identical ideas can be implemented using any other symbolic manipulation package or even the hand-held HP28S supercalculator.

Suppose we start with the problem of finding the area under a curve and introduce the notion of a Riemann sum for a function $f(x)$. Mathews describes how muMATH can be applied to obtain the limit of the Riemann sum in closed form, at least when $f(x)$ is a polynomial. In particular, he uses the muMATH function procedure

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FUNCTION RIEMANN (F, A,B,N)
     TERM:  EVSUM (F, X, A+J*(B-A)/N),
     TERM:  EXPAND (TERM),
     RSUM:  SIGMA (TERM, J, 1, N),
     RSUM:  RSUM*(B-A)/N
ENDFUN$
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to obtain the value for the Riemann sum for the function $f(x)$ on the interval $[A,B]$ with N subdivisions. For instance, if $f(x) = x^2$ on the interval [0,1] with $N = 100$ subdivisions, then we enter

? *F*: *x*^2;

? RIEMANN (*F*,0,1,100);

and within seconds, muMATH responds with a value for RSUM of 6767/20000. (Note that this has automatically been reduced to lowest terms.) Moreover, if N is used as a generic variable in the command

? RIEMANN (F,0,1, *N*);

then the muMATH system responds with the closed-form expression for RSUM in the form

1/3 + 1/(6*N*^2) + 1/(2 *N*).

This limit can now be evaluated in closed form either by hand or by using the muMATH limit operation as N approaches positive infinity applied to this expression:

LIM (RSUM, *N*, PINF);.

We therefore find that the limit of the Riemann sum is precisely 1/3.

In addition, it is possible to use muMATH to produce comparable results when an arbitrary interval $[A,B]$ is used instead of [0,1] as the limits of integration. In order to get the greatest effect out of this for a classroom demonstration, it is desirable to define the function f in terms of a dummy variable u by changing just one entry in the second line of the function definition above:

TERM: EVSUM (*F*, *U*, *A*+*J**(*B*−*A*)/*N*);.

Suppose that we use the function $f(u) = u^2$, and instead of using the interval [0,1], we use the interval $[0, x]$ for any arbitrary value of x. To do this, we enter the command

? *F*: *U*^2 ;

and have the system compute the Riemann sum based on N subintervals by entering

? RIEMANN (*F*,0, *X*,*N*);.

In this case, muMATH will respond with

X^3/(6 *N*^2) + *X*^3/(2 *N*) + *X*^3/3 .

If we now initiate muMATH's limit procedure

? LIM (RSUM, *N*, PINF);

the system responds with

$$X\verb|^|3/3.$$

That is, muMATH has established that

$$\int_0^x u^2\,du = x^3/3\,.$$

In a totally similar way using $f(u) = u^3$, we can find, in just a matter of seconds, that the Riemann sum on $[0,x]$ is given by

$$X\verb|^|4/(4\,N\verb|^|2) + X\verb|^|4/(2\,N) + X\verb|^|4/4,$$

and so, as N approaches infinity, the limit of the Riemann sum is precisely equal to X^ 4/4. Thus we have established that

$$\int_0^x u^3\,du = x^4/4\,.$$

From a pair of examples such as this, it is immediately apparent to the students that a clear pattern exists between the closed-form expression for the definite integral of a function $f(x)$ on the interval $[0, x]$ and the initial function itself. Based on this observation, there is a perfectly natural reason to introduce the notion of the antiderivative of an arbitrary function $f(x)$ as a means of expressing the definite integral. No longer is it necessary to make a totally unnatural digression to introduce the antiderivative and then several lectures later relate it to the definite integral.

In fact, the above derivation can be used to motivate much more of the theory involved in integration. Having performed the above motivation, it is now simple to introduce the need for an arbitrary constant of integration to generate the most general antiderivative of the function $f(x)$. Further, the First Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_0^x f(u)du = f(x)\,,$$

is immediately evident based on the above ideas. All that is needed, if one desires it, is to supply a proof for an arbitrary function $f(x)$.

In addition, it is also possible to extend the above ideas to consider the definite integral over an arbitrary interval $[A,B]$ with muMATH supplying the results in terms of A and B. Thus, if we repeat the above argument

RIEMANN (F, A,B,N);

with $f(u) = u^2$, then muMATH responds with the expression

$$(-A/N + B/N)(A\,B\,N/3 - A\,B/(3\,N) + A\verb|^|2\,N/3 + A\verb|^|2/(6\,N)$$
$$+ B\verb|^|2\,N/3 + B\verb|^|2/(6\,N) - A\verb|^|2/2 + B\verb|^|2/2),$$

for the Riemann sum. If we now apply the limit command to this,

? LIM (RSUM, *N* PINF);

we obtain

$$(-2A\hat{\ }3/3 + 2B\hat{\ }3)/6,$$

which immediately reduces to $B\hat{\ }3/3 - A\hat{\ }3/3$.

[We note that, in the process of obtaining this result, several intermediate inputs are necessary with muMATH. Since the system works in a totally symbolic mode, it does not have any direct way to determine whether any of the quantities involved, A and B, for instance, are positive or negative. Consequently, in order to evaluate the limit in closed form, muMATH will inquire of the user whether various terms are positive or negative.]

Consequently, the students clearly see that

$$\int_a^b u^2\,du = b^3/3 - a^3/3,$$

from which it is obvious to them that the value for this definite integral is given in the form $G(b) - G(a)$, where $G(x)$ is an antiderivative of $f(x)$. Thus, with one or two very quick examples, students immediately discover the Second Fundamental Theorem of Calculus entirely on their own. As before, all that is necessary is to couch the theorem formally in terms of an arbitrary function $f(x)$ and to supply the proof.

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REFERENCES

John H. Mathews, "Using Computer Algebra Systems to Teach the Fundamental Theorem of Calculus," *Math and Computer Education*, Vol. 23, pp. 199-204 (1989).

John H. Mathews, "Teaching Riemann Sums Using Computer Symbolic Algebra Systems," *College Mathematics Journal* (to appear).